New Ideas for Teaching Relativity with Minkowski Spacetime Diagrams

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Extended abstract:

We present two new approaches for teaching relativity[1] with Minkowski Spacetime Diagrams[2].

Visualizing Proper-Time:[3]

The first approach uses a new animated visualization of the proper-time elapsed along an observer's worldline. By supplementing inertial worldlines with light-clocks, we use the textbook storyline of the Michelson-Morley experiment[5] to construct the spacetime paths of light-rays associated with these light-clocks. These paths mark off intervals of proper-time in units of "ticks", where each tick is intuitively visualized as the intersection of the future light-cone of one tick-event with the past light-cone of the next tick-event. It can be shown that the spacetime volume enclosed by the tick is a Lorentz-invariant quantity. With the use of radar methods,[6,7,8,9,10,11] the measurements of spacetime intervals are then reduced to the "counting of ticks" The resulting space-time diagrams are pedagogically attractive because they emphasize the relativistic view that "proper-time is what is measured by an observer's clock." Although the details of the visualization require some analytic geometry, the physical ideas used in the construction are motivated qualitatively for a nontechnical target audience. We use the visualization to demonstrate the Clock Effect and the Doppler Effect. We conclude this section with an animated visualization for a uniformly-accelerated worldline.[12]

Spacetime Trigonometry:[4]

The second approach presents a unified formalism for two-dimensional Euclidean space, Galilean spacetime,[7,13] and Minkowski spacetime rooted in the geometrical studies of Arthur Cayley and Felix Klein.[14] Inspired by Yaglom[15] and Taylor and Wheeler,[16] we use familiar techniques from the analytic geometry and trigonometry of Euclidean space to develop the corresponding analogues for Galilean and Minkowski spacetimes and immediately provide them with physical interpretations. Upon defining a"unit circle" for each of the three geometries, we obtain line-elements that we write in a unified way as

$$ds^2 = dt^2 - \epsilon^2 dy^2,$$

where ϵ is an indeterminate quantity whose square [called the "indicator"] is -1, 0, or 1, corresponding to the Euclidean, [degenerate-] Galilean, or Minkowskian case. Physically, we define the indicator by $\epsilon^2 = (c_{light}/c_{signal,max})^2$, where $c_{signal,max} = \infty$ for the Galilean case, and $c_{signal,max} = c_{light}$ for the Minkowskian case. The notion of "orthogonality" is determined by tangents to the "circle". From there, the notions of "angle", "circular functions", and related constructions [including tensor visualization[11,17,18]] are developed. A feature of this unified formalism is the use of the indicator to clarify the analogies among the three geometries, especially the Galilean limits of Minkowskian results. In passing, we will also mention some connections with Euclid's Postulates and with hypercomplex number systems.[19] Although many of the facts presented here are known, they are scattered throughout the physics and mathematics literature, often with little reference to a common framework that is useful for the teaching of relativity. With this approach, we envision portions of a physics curriculum in which a student sequentially learns:

- Euclidean geometry and trigonometry, possibly with vector algebra;
- Galilean kinematics with the usual distance-vs-time plots (Galilean spacetime diagrams!), with at least some aspects of the underlying Galilean spacetime geometry;
- Special-Relativistic kinematics with Minkowski spacetime diagrams, highlighting geometrical and physical analogies (including the Galilean limits of Relativistic results) and further developing vector and tensor algebra using visualizations;
- introductory General-Relativistic kinematics using the DeSitter [and possibly Newton-Hooke] spacetimes and tensorial methods to further develop the Riemannian [rather than Kleinian] viewpoint to geometry and modern General Relativity.

[1] A. Einstein, "On the Electrodynamics of Moving Bodies" (1905) in H.A. Lorentz, A. Einstein, H. Minkowski, & H. Weyl, *The Principle of Relativity* (Dover Publications, New York, 1923), pp. 37-65.

[2] H. Minkowski, "Space and Time" (1909) in H.A. Lorentz, A. Einstein, H. Minkowski, & H. Weyl, *The Principle of Relativity* (Dover Publications, New York, 1923).

[3] R.B. Salgado, "Visualizing proper-time in Special Relativity", *Physics Teacher (Indian Physical Society)*, v.46 (4), pp. 132-143 (2004). The article is also available at: `http://arxiv.org/abs/physics/0505134`. The animated visualizations are available at `http://physics.syr.edu/courses/modules/LIGHTCONE/LightClock/`.

[4] R.B. Salgado, "Space-Time Trigonometry", *AAPT Topical Conference: Teaching General Relativity to Undergraduates*, Syracuse, NY (2006). `http://www.aapt-doorway.org/Posters/SalgadoPoster/SalgadoPoster.htm`.

[5] A.A. Michelson and E.W. Morley, "On the Relative Motion of the Earth and the Luminiferous Ether," *Am. J. Sci.*, 3rd Ser., 34, pp. 333-345 (1887).

[6] A.A. Robb, *Optical Geometry of Motion: a new view of the theory of relativity*, (Cambridge University Press, Cambridge, 1911).

[7] R.P. Geroch, *General Relativity from A to B* (University of Chicago Press, Chicago, 1978).

[8] G.F.R. Ellis and R.M. Williams, *Flat and Curved Space-Times* (Oxford University Press, Oxford, 1988).

[9] H. Bondi, *Relativity and Common Sense* (Dover, New York, 1962); H. Bondi, "Some Special Solutions of the Einstein Equations," in A. Trautman, F.A.E. Pirani, and H. Bondi, *Lectures on General Relativity: Brandeis 1964 Summer Institute on Theoretical Physics, vol. 1,* (Prentice-Hall, Englewood Cliffs, New Jersey, 1965), pp. 375-459.

[10] R.F. Marzke and J.A. Wheeler, "Gravitation as Geometry–I: The Geometry of Space-Time and the Geometrodynamical Standard Meter," in H.-Y. Chui and W.F. Hoffman (eds.), *Gravitation and Relativity*, (Benjamin, New York, 1964), pp. 40-64.

[11] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman, New York, 1973).

[12] A.D. Fokker, "Accelerated spherical light wave clocks in chronogeometry," *Nederl. Akad. Wetensch. Proc. Ser. B*, 59, pp. 451-454 (1956).

[13] R. Penrose, "Structure of Space-Time," in C.M. DeWitt and J.A. Wheeler (eds.), *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, (Benjamin, New York, 1967); A. Trautman, "Foundations and current problems of general relativity" in S. Deser and K.W. Ford (eds.), *Lectures on General Relativity.* (Prentice Hall, Englewood Cliffs NJ, 1965); D. Malament, "Newtonian Gravity, Limits, and the Geometry of Space", in Colodny, R. (ed.), *From Quarks to Quasars*, (University of Pittsburgh Press, Pittsburgh, 1986); J. Ehlers, "The Nature and Structure of Spacetime" in J. Mehra (ed.), *The Physicist's Conception of Nature*, (D. Reidel Publ, Dordrecht, Holland, 1973), pp. 71-91; H. Künzle, "Galilei and Lorentz structures on space-time: Comparison of the corresponding geometry and physics", *Annales Institute Henri Poincaré*, 17, pp. 337-362 (1972).

[14] A. Cayley, "A sixth memoir upon quantics", *Phil. Trans. R. Soc., London*, 149, pp. 61-90 (1859); F. Klein, "A comparative review of recent researches in geometry", *Bull. Amer. Math. Soc.*, 2 , pp. 215-249 (1893); D.M.Y. Sommerville, "Classification of geometries with projective metric", *Proc. Edinburgh. Math. Soc.*, 28, pp. 25-41 (1910).

[15] I.M. Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis*, (Springer-Verlag, Berlin, 1979) (Russian original 1969); G.G. Emch, *Mathematical and conceptual foundations of 20th-century physics*, (North-Holland Publishing Co., Amsterdam, 1984).

[16] E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, (W.H. Freeman, New York, 1966).

[17] J.A. Schouten, *Der Ricci-Kalkül*, (Springer, Berlin, 1924); J.A. Schouten, *Ricci Calculus*, (Springer, Berlin, 1954); J.A. Schouten, *Tensor Analysis for Physicists* (2nd edition), (Dover Publications, New York, 1989).

[18] W.L. Burke, *Applied Differential Geometry*, (Cambridge University Press, Cambridge, 1985); B. Jancewicz, *Multivectors and Clifford Algebra in Electrodynamics*, (World Scientific, Singapore, 1988).

[19] I.M. Yaglom, *Complex Numbers in Geometry*, (Academic Press, New York, 1968); I.L. Kantor and A.S. Solodovnikov, *Hypercomplex Numbers: An Elementary Introduction to Algebras*, (Springer, Berlin, 1989).