The Metrics of General Relativity and Quantum Mechanics

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Abstract
There is a fundamental paradigm shift between general relativity and classical mechanics, characterized by the fact that in general relativity the energy-momentum tensor is the effective cause of the ontological space-time curvature and vice-versa, while in classical physics, the structure of space-time is treated as an accidental cause, serving only as a backdrop against which the laws of physics unfold. This split in turn is inherited by quantum mechanics, which is usually developed by changing classical (including special relativity) Hamiltonians into quantum wave equations. In this paper, we try to remedy this situation by taking the (locally Minkowski) metrics of general relativity as the starting point of quantum mechanics. We will associate wave equations in a natural way with those operators which are duals of differential one-forms (expressed locally as a Minkowski metric) rather than with operators derived from a Hamiltonian, thus enabling the ontological structure of space-time itself to determine in a natural and unique way the wave equations of quantum mechanics.

Specifically, at every point p on the space-time manifold $\mathcal{M}$ we erect a local tetrad $e_0(p), e_1(p), e_2(p), e_3(p)$ such that a point x has coordinates $x = (x^0,\ x^1,\ x^2,\ x^3) = x^a e_a$ in this tetrad coordinate system, while the spinor ψ can be written as $\psi = \psi^i e_i(p)$, where ψ^i represent the coordinates of the spinor with respect to the tetrad at p. Also at p we can establish a tangent vector space $T_p(\mathcal{M})$, with basis $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ and a dual 1-form space, denoted by T_p^* with basis $\{dx_0, dx_1, dx_2, dx_3\}$ at p, defined by

$$dx^\mu \partial_\nu \equiv \partial_\nu x^\mu = \delta^\mu_\nu. \tag{1}$$

We refer to the basis $\{dx^0, dx^1, dx^2, dx^3\}$ as "the basis of one forms dual to the basis $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ of vectors at p."

Now consider the metric equation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b \tag{2}$$

where a and b refer to local tetrad coordinates and η to a rigid Minkowski metric of signature -2, then associated with this metric and the vector **ds** is the scalar ds and a matrix $\tilde{ds} \equiv \gamma_a dx^a$ respectively, where $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, with γ_a transforming as a covariant vector under coordinate transformations. Note also that $g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x)$ with $e^a_\mu(x)$ forming local tetrads at x. Moreover, since ds is an invariant scalar, and $\tilde{ds}^2 = ds^2$ we can identify the "eigenvalue" ds with the linear operator $\tilde{ds}$ by forming the spinor eigenvector equation $\tilde{ds}\xi = ds\xi$. This is equivalent to associating the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b \tag{3}$$

with the spinor equation:

$$ds\xi = \gamma_a dx^a \xi. \tag{4}$$

As previously noted, corresponding to each tangent vector $\frac{\partial}{\partial x^a}$, there exists a dual one-form dx^a. In a similar way, the $\tilde{ds}$ matrix above can be seen as the dual of the expression $\tilde{\partial}_s \equiv \gamma^a \frac{\partial}{\partial x^a}$, where γ^a is defined by the relationship $\{\gamma^a, \gamma_b\} = 2\delta^a_b$ and the dual map defined by

$$\left\langle \tilde{ds}, \tilde{\partial}_s \right\rangle \equiv \frac{1}{\mathrm{Tr}(dx^i \partial_j)} \gamma_a \gamma^b dx^a \frac{\partial}{\partial x^b} \equiv \frac{1}{\delta^i_i} \gamma_a \gamma^b \frac{\partial x^a}{\partial x^b} = 1, \tag{5}$$

remains invariant. Moreover, if we let s describe the length of a particle's trajectory along a geodesic $(x^0(s), x^1(s), x^2(s), x^3(s)) \in (\mathcal{M}, g)$ then s can be regarded as an independent parameter with an associated 1-form ds, which is the dual of the tangent vector ∂_s. Note that in terms of the basis vectors for $T_p(\mathcal{M})$ and $T^*_p(\mathcal{M})$ we can write $\partial_s = \frac{\partial x^a}{\partial s}\partial_a$ and $ds = \frac{\partial s}{\partial x^a} dx^a$. It also follows from this and equation (1) that its dual map is given by $ds.\partial_s = \frac{\partial s}{\partial x^i}\frac{\partial x^i}{\partial s} = 1$. Putting these two results together allows us to consider equation (4) as the dual of the equation:

$$\frac{\partial \psi}{\partial s} = \gamma^a \frac{\partial \psi}{\partial x^a}, \tag{6}$$

where $\frac{\partial}{\partial s}$ refers to differentiation along a geodesic parametrized by s. We will refer to (6) as a (generalized) Dirac equation and will show how it relates to the usual form of this equation by applying it to the Minkowski metric (potential well problem), the schwartzschild metric, and the Reissner-Nordstrom metric (the hydrogen atom). We will also show how classical mechanics can be incorporated into this framework by considering point-mass solutions of the Dirac equation, derived from the Minkowski metric. Indeed, it would seem that the local Minkowski structure of space-time determines the laws of mechanics in general.