The Equation of State of Space in Minkowski Spacetime

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It can be shown that if the "empty" spacetime of Minkowski is assumed to have intrinsic energy then basic principles of mechanics and special relativity require that the pressure of this spacetime must equal its energy density. Consider a volume element of Minkowski spacetime (MST) in a region of space that is well removed from gravitational influences. For the sake of visualization, imagine this spatial volume element to be enclosed in a box that has been completely evacuated of all known forms of matter and radiation. Now, consider an observer, initially at rest relative to the volume element of MST, who begins to accelerate past it. We know from relativity theory that, relative to the observer, our volume element of the MST will contract and its energy will increase with increasing speed. More precisely, if a volume element of MST having volume dV_0 and energy dE_0 relative to an observer at rest is observed from a reference frame that is moving with relative speed v, its instantaneous volume relative to such an observer will be:

$$dV = \left[1-\left(v^2/c^2\right)\right]^{1/2} dV_0, \tag{1}$$

and its instantaneous energy will be:

$$dE = \left[1-\left(v^2/c^2\right)\right]^{-1/2} dE_0. \tag{2}$$

If the pressure of the MST is not assumed to be zero, then basic principles of mechanics require that the work done by that pressure during a small, adiabatic contraction of the volume be related to the corresponding change in the energy of the element by: $d(dE) = -pd(dV)$. That is, in order for the work-energy theorem of mechanics to remain valid for an accelerating observer, the gain in energy of our volume element must be explained locally by the work done on it by the pressure of the surrounding space. While the work-kinetic energy theorem of particle mechanics permits us to relate work done by net external forces on a point mass in a vacuum (presumed to be at zero pressure) to changes in its kinetic energy, a new conceptual paradigm is necessitated in relativistic continuum mechanics by the fact that the pressure may not generally be assumed zero, and so must be considered to do work on an element of the medium undergoing volume change due to Lorentz contraction. The pressure of our volume element of MST must therefore be calculated as the negative derivative of the *total* energy of the element with respect to the volume of the element in order for the work-energy theorem to remain a valid, Lorentz-invariant expression of energy conservation.

In order for the energy and volume of our element of the MST to change relative to an accelerating observer in accordance with (1) and (2), as well as conserve energy locally, the pressure must be

$$p = -\frac{d(dE)}{d(dV)} = -\frac{d(dE)/dv}{d(dV)/dv}. \qquad (3)$$

Or, upon taking the derivatives,

$$p = \frac{dE_o}{dV_o[1-(v^2/c^2)]} = \frac{dE}{dV} \equiv e. \qquad (4)$$

We see that the equation of state of the MST is simply $p = e$.

It is instructive to discuss here some of the implications of this equation of state. Clearly, the energy described by this equation of state is quite unlike any bulk matter or energy observed in the laboratory. For example, in the equation of state for an ultra relativistic ideal gas, the pressure approaches $\frac{1}{3}e$ as the mean thermal energy of the particles becomes much greater than their rest energy. We see that an element of "space-energy" has three times the pressure, for a given energy density, as an ultra relativistic ideal gas. It is also apparent that this space-energy has the greatest possible proportionality coefficient, w, in an equation of state of the form $p = we$. The speed of sound in a relativistic medium is equal to c times the square root of the adiabatic derivative of the pressure with respect to the energy density. Since this derivative is unity for the equation of state $p = e$, the speed of sound in the energy field is equal to the speed of light. Therefore no greater value of w would be consistent with basic principles of relativity and causality, since that would result in a signal speed greater than c.

An intensive measure of a medium's resistance to being compressed is its bulk modulus, $B \equiv r\left(dp/dr\right)_s$ where r is the mass density and the derivative is taken at constant entropy. Using Einstein's mass-energy equation in the form $e = rc^2$, we find that $B = rc^2 = p$. Therefore, the compressibility, defined as the inverse of the bulk modulus, is zero in the non-relativistic limit $c \to \infty$. The space-energy field, as described by our equation of state, is therefore the proper relativistic generalization of the non-relativistic, incompressible fluid of Newtonian physics.

The paradigm shift employed to derive our equation of state also provides us with an interesting new conceptual perspective on the nature of inertial mass. Just as work must be done on ordinary matter to compress it, work would have to be done on a mass element of space to force it to undergo Lorentz contraction when it is accelerated. As is readily seen from the previously derived relationships, the mass of an element of space is given by: $m = BV/c^2$. Thus the mass of an element of space is directly proportional to its bulk modulus, which is its resistance (per unit volume) to being compressed.