# Physics and the Local Mathematical Structure of Space-Time

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**Abstract:**

The quantitative data-set of experimental physics (and of observational astronomy) consists, at any instant, of a finite collection of finite sets of rational numbers. Theoretical physics is based on the assumption that space and time (or space-time) are *continua*. This assumption cannot be "tested" by any finite set of observations. As a result, there is a gap between experimental and theoretical physics, which does not seem to have attracted much attention in the community of theoretical physicists.

It would therefore be of interest to examine, if possible, the relation between physics and the mathematical structure of space-time by purely mathematical means. If we make the assumption that *it makes sense to use the Euclidean notion of geometrical points in physics*, then it is indeed possible to carry out such an investigation. The point of departure is provided by the realization (or discovery) that there exists a purely physical principle—namely, that no signal can propagate faster than light—which can be defined on a point-set devoid of mathematical structure, and, once defined, determines a great deal of mathematical structure on the point-set. We shall call this principle *Einstein-Weyl* (or E-W) *causality.*

On Minkowski space, E-W causality is a partial order (the past-future order). The first problem, therefore, is to axiomatize this partial order on a point-set $M$ that has no predefined mathematical structures on it. Happily, it turns out that one can do so by axiomatizing the notion of *light rays* (by a light ray we mean the spacetime path of a light ray in geometrical optics) and their properties. Light rays are subsets of $M$ that are totally ordered by the past-future relation.

Our scheme is based on four axioms, two of which provide structure and two that rule out excessive generality. An axiom called the *order axiom* defines light rays and their properties. The latter include the important *density property*, which states that between any two distinct points on a light ray lies

a third. A further axiom called the *local structure axiom* defines, in terms of light rays and their intersections, the structure of a "neighbourhood" of any point in $M$, and requires every point of $M$ to have such a neighbourhood. The axioms that rule out excessive generality (such as closed timelike curves) nevertheless admit separable and nonseparable Hilbert spaces. The structure defined on $M$ by these axioms is called, for brevity, an *order structure*, and the space $M$ an *ordered space*.

As mentioned above, the order structure has far-reaching mathematical consequences. It defines, firstly, 1) a Hausdorff topology (named the *order topology*), and then 2) a separable uniformity on $M$. It is a standard result in point-set topology that a uniform space $M$ has a uniform completion $\tilde{M}$, and that $M$ is dense in $\tilde{M}$. It turns out that 3) the order structure of $M$ can be extended to $\tilde{M}$.

If the space $\tilde{M}$ is locally compact and finite dimensional, it follows, from standard results, that 4) $\tilde{M}$ has a differentiable structure. However, examples show that 5) the order structure of $\tilde{M}$ need not be differentiable, i.e., light rays need not be differentiable curves. One needs further assumptions to ensure that the order structure is differentiable. A sufficient condition for this is 6) that spacetime be locally isotropic. This assumption also implies 7) a Minkowski structure on the tangent space $T_x$ at any point $x \in \tilde{M}$.

The above is a summary of the results obtained so far on the mathematical structure of spaces satisfying E-W causality. We note that the density property of light rays, which, rather trivially, forces each light ray and therefore the space $M$ to contain infinitely many points, falls short of forcing $M$ to be a continuum. For example, the space $Q^2$ becomes an ordered space if light rays are defined to be straight lines at 45° with the axes. However, just as $Q^2$ is dense in $R^2$, the space $M$ is dense in its uniform completion $\tilde{M}$. And, at least in the locally-compact finite-dimensional case, $\tilde{M}$ carries a differentiable structure. While E-W causality fails to force a differentiable structure, spaces that satisfy it are densely embedded in differentiable manifolds.

Referring to Wigner's famous address "On the unreasonable Effectiveness of Mathematics in the Natural Sciences", one may ask the following, rather more limited question: Is the differential calculus a discovery or an invention? Since it is not possible to distinguish, by experiment, between causal $Q^2$ and causal $R^2$, the question posed above would appear to be *undecidable* within the framework of the scientific method.[1]

[1]The reservation *would appear to be* is made advisedly. In view of the finite precision of

## References


1. Alexandrov, A. D., Filosofskoe soderzhanie i znachenie teorii otnositel'nost, *Voprosy Filosofii* **1**, 67 (1959). (Alexandrov was apparently the first to notice that the interiors of double cones in Minkowski space formed a base for the usual topology of the space. However, he did not notice that, in Newtonian space-time, the interiors of double cones gave rise only to a non-Hausdorff topology.)

2. Borchers, H.-J. and R. N. Sen, Theory of Ordered Spaces, *Commun. Math. Phys.* **132**, 593-611 (1990). (The order topology was introduced here.)

3. Borchers, H.-J. and R. N. Sen, Theory of Ordered Spaces, II. The local differential structure, *Commun. Math. Phys.* **204**, 475-492 (1999). (This paper established that second countable locally compact ordered spaces, with the additional assumption that light rays were locally homeomorphic to $R$, had a local differentiable structure.)

4. Borchers, H.-J. and R. N. Sen, *Mathematical Implications of Einstein-Weyl Causality*, to appear in *Lecture Notes in Mathematics*, Springer-Verlag. (The result that the order structure defines a Hausdorff uniformity, and its consequences, will appear in these Notes, which will give a coherent—and hopefully readable—account of the entire work.)

5. Sen, R. N., Why is the Euclidean Line the same as the Real Line?, *Foundations of Physics Letters* **12**, 325-345 (1999). (From the Abstract: "This paper investigates whether the differential structure of spacetime follows from accepted laws of physics, or is a mathematical invention.")

6. Wigner, E. P., The Unreasonable Effectiveness of Mathematics in the Natural Sciences, *Comm. Pure and Appl. Math.* **13**, 1 (1960). (Wigner accepted Cantor's dictum that "Mathematics is the free creation of the human mind". Not being burdened with Cantor's theological preoccupations, he could see the problem in stark relief. He assumed that it


quantitative measurements, the use in physics of the notion of geometrical points remains to be justified. This is relatively straightforward in classical physics (although the scheme devised by the author may not be unique), but less so in quantum physics, largely because of controversies concerning the theory of measurement in quantum mechanics. A coherent answer to all these questions is currently under preparation.

is possible to distinguish between a discovery and an invention. Some philosophers may disagree.)