Ontology of Quantum Space interpreted by Quantum Real Numbers.

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Abstract

Intuitionistic real numbers are constructed as sheaves on the state space of the Schrodinger representation of a CCR-algebra with a finite number of degrees of freedom. These numbers are used as the values of position and momentum variables that obey Newton's equations of motion. Heisenberg's operator equations of motion are shown to give rise to numerical equations that, on a family of open subsets of state space, are local approximations to Newton's equations of motion for the intuitionistically valued variables.

The ontology of space in quantum mechanics can only be understood through a viable theory of quantum mechanics. A viable theory is a theory that is complete in the sense described by Einstein, "A complete system of theoretical physics is made up of concepts, fundamental laws which are supposed to be valid for those concepts and conclusions to be reached by logical deduction. It is these conclusions which must correspond with our separate experiences;....." [4]. It can be argued that the standard theory of quantum mechanics is not complete in this sense because some of the separate experiences, e.g. those of measurement, that the theory should be able to reach by logical deductions from its fundamental laws and concepts are not obtainable within the theory [3]. Moreover, quantum mechanics has conclusions corresponding to experiences, such as the two slit experiment, that present severe interpretative difficulties. There is general agreement that most of the "paradoxes" of quantum mechanics depend upon "the relation between - on the one hand - the values of physical quantities, and - on the other hand - the results of measurements" [6]. We claim that a resolution of these interpretative problems is found by constructing for the ontological values of physical quantities a real number continuum that generalises and contains the standard real number continuum.

In this approach, the real number continuum is no longer taken to be an absolute element of the theory, its structure will depend upon the physical system being described. This dependence of the structure of the real number continuum on the physical situation is analogous to the dependence of the metrical structure of space-time in general relativity on the physical situation that it describes.

It is a mathematical fact, see for example [7] section VI.8, that the standard real number continuum gives us the prototype of a class of real number continua in much the same way as Euclidean geometry provides the prototype of the class of Riemannian geometries. The class of real number continua is called the Dedekind reals R_{D} given by the sheaves of continuous real valued functions on a topological space X. To obtain the standard reals just take the topological space X to be the one point space. For quantum mechanical systems, we take the real number continuum to be the continuum of Dedekind reals R_{D} in the topos $\mathrm{Shv}(\mathcal{E}_{\mathcal{S}})$ of sheaves on the topological space $\mathcal{E}_{\mathcal{S}}$, where $\mathcal{E}_{\mathcal{S}}$ is the set of all normalised, strongly positive linear functionals (states)on the *Schrödinger* representation of the algebra generated by the canonical commutation relations [2]. The topology on $\mathcal{E}_{\mathcal{S}}$ is chosen to make continuous all the functions of the form $a_Q(\hat{\rho}) = Tr(\hat{\rho} \cdot \hat{A}) : \hat{\rho} \in \mathcal{E}_{\mathcal{S}}$. Then the functions a_Q form a subobject A of R_{D} on $\mathcal{E}_{\mathcal{S}}$ that we call the quantum real numbers. Because the logic of the topos $\mathrm{Shv}(\mathcal{E}_{\mathcal{S}})$ is non-Boolean intuitionistic [5], some of the mathematical properties of the quantum real numbers differ from those of standard real numbers [2], [8].

All the mathematical ingredients used in the construction of the Dedekind real number continuum R_{D} in the topos $\mathrm{Shv}(\mathcal{E}_{\mathcal{S}})$ are present in the standard Hilbert space formalism of quantum mechanics[9].

A physical interpretation of these non-standard real numbers may be linked, via the interpretation of a quantum state $\hat{\rho}$ as representing a preparation process, to the requirement that the whole experimental arrangement must be included in the determination of physical quantities. However our model differs from this requirement in two important ways: firstly, the physical processes that constitute the preparation process may occur naturally without the intervention of experimentalists and, secondly, the preparation procedures are represented by open sets of states. For example, the quantum real number value $a_Q(W)$ of a quantity that is represented in standard quantum theory by the operator $\hat{A}$ is given by a function labelled by $\hat{A}$ and defined on a domain W that is an open subset of state space. The open subset of state space represents an equivalence class of preparation processes that give the quantity that quantum real number value. In the category of sheaves on $\mathcal{E}_{\mathcal{S}}$, the open set W is a truth value, the extent to which the quantum real number value of the physical quantity is held. We assume that physical quantities always have numerical values expressible as quantum real numbers or as functions of quantum real numbers but do not always have standard real number values.

The cartesian product ${R_{\mathrm{D}}}^3$ of three copies of the Dedekind real numbers R_{D} is taken to be the three space of quantum mechanics. Its geometric properties depend upon the structure of the underlying continuum of R_{D} [8]. R_{D} is a field that is partially ordered but not totally ordered, in particular, trichotomy does not always hold for quantum real numbers. Also it is not Archimedean in the sense that there are numbers $a \in R_{\mathrm{D}}$ such that $a > 0$ but there is no natural number $n \in N$ such that $n > a$. R_{D} is a complete metric space with respect to a distance function derived from the norm function $|\cdot| : R_{\mathrm{D}} -- > R_{\mathrm{D}}$ that takes a to $max(a, -a)$ [8]. With this we can define a distance function between points in ${R_{\mathrm{D}}}^3$ which can be used to define quantum localisation of quantum particles.

Quantum localisation is different from classical localisation. For example, suppose that the z-coordinate of a particle has the quantum real number value $z_Q(W)$, where W is the union of disjoint open subsets, $W = U \cup V$ with $U \cap V = \emptyset$. $z_Q(W)$ is a single quantum real value and hence represents a single point on the z-axis in quantum space. But if there are standard real numbers $z_1 < z_2 < z_3 < z_4$, with $z_1 < Z_Q(U) < z_2$, $z_3 < Z_Q(V) < z_4$ and $z_2 \ll z_3$ then the single quantum real number value may be viewed classically as a pair of separated intervals of standard real numbers and hence non-localised. This example of the difference between quantum and classical localisation can be used to understand the 2-slit experiments. A particle may be localised in terms of the quantum real number values of its position but not localised in terms of the classical standard real number values. Similarly the quantum distance between a pair of particles may be small even though the classical distance between them is large.

According to this theory, the ontology of quantum particles is the same as that of non-relativistic particles of classical mechanics except that the numerical values of their positions and momenta are given by quantum real numbers. The dynamical equations of motion of a quantum particle are given by Hamilton/Newton's equations expressed in quantum real numbers. Thus in terms of the quantum real number values of physical quantities the theory is deterministic, but not necessarily deterministic if we seek to connect standard real number values of quantities. Because Heisenberg's operator equations of motion when averaged over certain open subsets of state space closely approximate the Newtonian equations for quantum real numbers defined on these open sets, they can be interpreted as empirically adequate dynamical laws that locally approximate the basic quantum real number dynamics.

The Dedekind reals R_{D} contain copies of the standard reals R and copies of the standard rationals Q as constant functions. This has important consequences in the measurement problem. In a measurement a standard real number is sifted from the existing quantum real number, however a quantum real number could give rise to many different standard real numbers. To obtain a unique standard real number value we must use a sufficiently fine classical grating $G = \{E_1, E_2, ..., E_n\}$ where the E_j satisfy $E_j^2 = E_j$ and $E_j \cdot E_k = 0$ if $j \neq k$ [10]. When the sifting operations E_j determine strictly ϵ sharp collimated quantum real numbers the Luders-von Neumann transformation (collapse of the wave packet) rule provides a good approximation to the sifted quantum real numbers. There is a different way of posing the measurement problem. Do there exist Newtonian-type forces that when expressed in quantum real numbers allow a quantity whose quantum real number values are not well approximated by a single standard real number to evolve so that its quantum real number values do become well approximated by a single standard real number ? It seems reasonable that such forces exist, but we do not yet have an answer to this question.

References

[1] M.Adelman and J. V. Corbett: "A Sheaf Model for Intuitionistic Quantum Mechanics", Applied Categorical Structures **3** 79-104 (1995)

[2] M.Adelman and J.V. Corbett: "Quantum Mechanics as an Intuitionistic form of Classical Mechanics" to appear in Proceedings of the Centre Mathematics and its Applications, ANU, Canberra (2001).

[3] J.S.Bell, *Speakable and Unspeakable in Quantum Mechanics"*, pp117-118 (Cambridge University Press,1993)

[4] A. Einstein, " On the Method of Theoretical Physics", The Herbert Spencer lecture, delivered at Oxford University, June 10, 1933. Republished in "Ideas and Opinions", pp264-270Crown Publishers, New York, (1954).

[5] A. Heyting: "Intuitionism, an introduction" eds. North Holland, Amsterdam.(1971).

[6] C.J.Isham,J.Butterfield,"Some possible Roles for Topos Theory in Quantum Theory and Quantum Gravity",Fndns of Physics (1999).

[7] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic* (Springer–Verlag, New York, 1994).

[8] L. N. Stout, Cahiers Top. et Geom. Diff. **XVII**, 295 (1976); C. Mulvey, "Intuitionistic Algebra and Representation of Rings" in Memoirs of the AMS **148** (1974).

[9] J. V. von Neumann: "Mathematische grundlagen der quanten-mechanik", Springer-Verlag, Berlin (1932).English translation: "Mathematical Foundations of Quantum Mechanics", Princeton University Press, Princeton, 1955.

[10] H. Weyl: "Philosophy of Mathematics and Natural Science" Appendix C, pp255-257, Princeton University Press, Princeton (1949).